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The Upper Bound of K in K-Lossless Sequential Machines

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A sequential machine is said to be k-lossless if k is the least integer such that the initial state and the first k outputs uniquely determine the initial input. For n-state machines the upper bound of k is $\frac{1}{2}n(n-1)+1$ and the lower bound of k is 1. In this paper it will be shown that there exists an n-state $\{\frac{1}{2}n(n-1)+1\}$-lossless sequential machine with 3 input symbols and 6 output symbols. This problem is related to the computation capability of n-state sequential machines.

1. INTRODUCTION

A sequential machine is said to be k-lossless if k is the least integer such that the initial state and the first k outputs uniquely determine the initial input. A necessary and sufficient condition that a sequential machine is k-lossless is given by Even (1965). Using his result it can be shown that the upper bound of k is $\frac{1}{2}n(n-1)+1$ for n-state machines, but a k-lossless machine satisfying this bound was not obtained. In this paper it will be shown that there exists an n-state sequential machine which meets the upper bound for any n.

A Mealy-type sequential machine M can be defined as a system consisting of S, X, Y, δ, and λ, i.e., $M=(S, X, Y, \delta, \lambda)$, where S is the set of a finite number of states, i.e., $S=\{s_1, s_2, \ldots, s_n\}$; X is the set of a finite number of input symbols, i.e., $X=\{x_1, x_2, \ldots, x_p\}$; Y is the set of a finite number of output symbols, i.e., $Y=\{y_1, y_2, \ldots, y_q\}$; and δ is the next state function and λ is the output function satisfying the following equations:

$$s(t+1)=\delta(s(t), x(t)), \tag{1}$$

$$y(t)=\lambda(s(t), x(t)), \tag{2}$$

where $s(t)$, $x(t)$ and $y(t)$ are the state, the input, and the output at time t (which is an integer).

DEFINITION 1 [Huffman (1959)]. A sequential machine is said to be k-lossless if k is the least integer such that the initial state and the first k outputs uniquely determine the initial input; that is,

$$x(t-k) = g(y(t-1),\dots, y(t-k), s(t-k)). \tag{3}$$

If a serial connection of M and M' corresponds a delay line of length h, M' is said to be a quasi inverse of M of delay h. For any k-lossless machine M there exists a quasi inverse machine M' of delay $h \geqslant k-1$. A quasi inverse machine of delay 0 is said to be an inverse machine.

A sequential machine is said to be information lossless if every time the input sequence is determined by the initial state, the final state, and the output sequence. Obviously, a k-lossless machine is information lossless. However, it is known that there exists an information lossless sequential machine without any quasi inverse sequential machines [Huffman (1959)].

2. THE UPPER AND LOWER BOUNDS OF k OF k-LOSSLESS MACHINES

In this section the upper and lower bounds of k in n-state k-lossless machines are discussed.

DEFINITION 2. The pair graph $G_0(M)$ of the sequential machine M is composed of the following:

(1) A node for each unordered pair $(s_i, s_j) = (s_j, s_i)$.

(2) If $\delta(s_i, x) = s_k$, $\delta(s_j, x') = s_h$, $\lambda(s_i, x) = y$, and $\lambda(s_j, x') = y$, where x and x' may be identical, there is a directed arc from the node (s_i, s_j) to the node (s_k, s_h).

THEOREM 3. *A sequential machine M is information lossless if and only if the following conditions are satisfied* [*Even* (1965)].

(1) *For any h, define a pair of states (s_{ih}, s_{jh}), where $s_{ih} \neq s_{jh}$ are defined for every s_h such that*

$$\begin{gathered} x, x' \in X, \qquad s_{ih}, s_{jh}, s_h \in S, \qquad x \neq x', \\ s_{ih} = \delta(s_h, x), \\ s_{jh} = \delta(s_h, x'), \\ \lambda(s_h, x) = \lambda(s_h, x'). \end{gathered}$$

(2) *The graph $\tilde{G}_0(M)$ obtained from $G_0(M)$ by deleting the nodes which are not accessible from (s_{ih}, s_{jh}) for any h contains no nodes of repeated states, i.e., (s_i, s_i).*

THEOREM 4. *A sequential machine M is k-lossless if and only if the following conditions are satisfied* [*Even* (1965)]:

(1) *M is information lossless.*

(2) *The graph $\tilde{G}_0(M)$ contains no directed circuits and the longest directed path contains $k-1$ nodes.*

As there are $\frac{1}{2}n(n-1)$ nodes in $G_0(M)$ for n-state machine M, the following corollary is obtained.

COROLLARY 5. *If an n-state sequential machine is k-lossless, k satisfies* (4):

$$1 \leqslant k \leqslant \tfrac{1}{2}n(n-1) + 1. \tag{4}$$

We can show that these bounds are tight by constructing such machines.

THEOREM 6. *For any n there exists a $\{\frac{1}{2}n(n-1)+1\}$-lossless n-state sequential machine with 3 input symbols and 6 output symbols.*

Proof. We show that the machine M defined by Table I is $\{\frac{1}{2}n(n-1)+1\}$-lossless. In the table, $[x]$ means the largest integer not exceeding x. For

TABLE I

Input	0		1		2	
1	2,	0	3,	1	2,	4
2	3,	0	4,	1	3,	4
⋮	⋮	⋮	⋮	⋮	⋮	⋮
i	$i+1$,	0	$i+2$,	1	$i+1$,	4
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$[n/2]-1$	$[n/2]$,	0	$[n/2]+1$,	1	$[n/2]$,	4
$[n/2]$	$[n/2]+1$,	0	1,	3	2,	3
$[n/2]+1$	$[n/2]+2$,	0	$[n/2]+2$,	2	$[n/2]+2$,	4
⋮	⋮	⋮	⋮	⋮	⋮	⋮
j	$j+1$,	0	$j+1$,	2	$j+1$,	4
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$n-2$	$n-1$,	0	$n-1$,	2	$n-1$,	4
$n-1$	n,	0	n,	2	n,	4
n	1,	1	1,	2	1,	5
Present state	Next state, Output					

simplicity, s_i is denoted by i. Obviously, $(1, 2)$ is the only node satisfying condition (1) of Theorem 3. As there is no state pair (i, j) $(i \neq j)$ satisfying

$$\exists x, x' \in X, \qquad \begin{aligned} \delta(i, x) &= \delta(j, x'), \\ \lambda(i, x) &= \lambda(j, x'), \end{aligned}$$

there exists no edge from any node (i, j) $(i \neq j)$ to any node of repeated states. Thus, the machine M is information lossless. The remaining parts of the proof are as follows:

(1) There exists a path containing all nodes and initiated from node $(1, 2)$ in $G_0(M)$.

(2) There exist no loops in $G_0(M)$.

Edges in the pair graph $G_0(M)$ are classified into following types:

[Type I] Edges corresponding to outputs 0 and 4, i.e.,

$$\delta(i_1, x) = i_2, \qquad \delta(j_1, x') = j_2,$$
$$\lambda(i_1, x) = \lambda(j_1, x') = 0 \qquad (\text{or } 4).$$

This type of edges connects nodes as

$$(i, j) \to (i + 1, j + 1) \qquad (i < j < n).$$

The number of edges of this type is

$$\frac{(n-1)(n-2)}{2}.$$

[Type II] Edges corresponds to output 1, i.e.,

$$\delta(i_1, x) = i_2, \qquad \delta(j_1, x') = j_2,$$
$$\lambda(i_1, x) = \lambda(j_1, x') = 1.$$

This type of edges connects nodes as

$$[\text{II-a}] \quad (i, n) \to (1, i + 2) \qquad \left(i < \left[\frac{n}{2}\right]\right),$$

$$[\text{II-b}] \quad (i, j) \to (i + 2, j + 2) \qquad \left(i < j < \left[\frac{n}{2}\right]\right).$$

The number of edges of [II-a] is

$$\left[\frac{n}{2}\right] - 1$$

[Type III] Edges corresponding to output 2, i.e.,

$$\delta(i_1, x) = i_2, \qquad \delta(j_1, x') = j_2,$$
$$\lambda(i_1, x) = \lambda(j_1, x') = 2.$$

This type of edges connects nodes as

$$\text{[III-a]} \quad (i, n) \to (1, i+1) \qquad \left(\left[\frac{n}{2}\right] < i < n\right),$$

$$\text{[III-b]} \quad (i, j) \to (i+1, j+1) \qquad \left(\left[\frac{n}{2}\right] < i < j < n\right).$$

The number of edges of [III-a] is

$$\left[\frac{n+1}{2}\right] - 1.$$

Any edge in the graph belongs to one of the above classes. We can show that the edges of types I, II-a and III-a form a path containing all nodes (i, j) $(i \neq j)$, that is,

$$\begin{array}{ll}
(1, 2) \to (2, 3) \to \cdots \to (n-1, n) & \text{(Type I)} \\
(n-1, n) \to (1, n) & \text{(Type III-a)} \\
(1, n) \to (1, 3) & \text{(Type II-a)} \\
(1, 3) \to (2, 4) \to \cdots \to (n-2, n) & \text{(Type I)} \\
(n-2, n) \to (1, n-1) & \text{(Type III-a)} \\
(1, n-1) \to (2, n) & \text{(Type I)} \\
(2, n) \to (1, 4) & \text{(Type II-a)} \\
(1, 4) \to (2, 5) \to \cdots \to (n-3, n) & \text{(Type I)} \\
\vdots & \\
\left(1, \left[\frac{n+1}{2}\right] + 1\right) \to \cdots \to \left(n - \left[\frac{n+1}{2}\right], n\right) & \text{(Type I)}
\end{array}$$

Such a path for $n = 8$ is shown in Fig. 1. As the total number of edges in these types is

$$\frac{(n-1)(n-2)}{2} + \left[\frac{n}{2}\right] - 1 + \left[\frac{n+1}{2}\right] - 1 = \frac{1}{2}n(n-1) - 1,$$

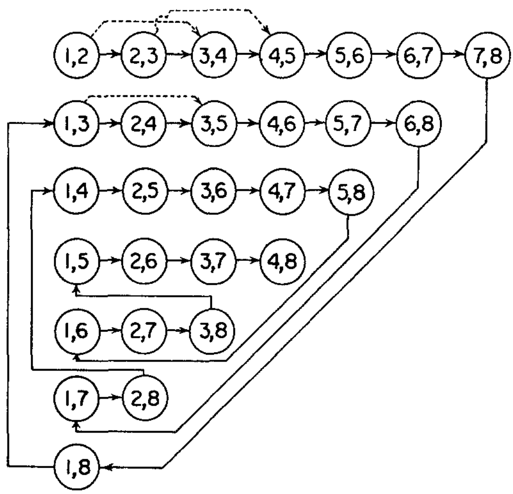

FIG. 1. A pair graph of a 29-lossless 8-state sequential machine.

all the edges are contained in this path. The remaining edges are types II-b and III-b. As every edge of type II-b connects nodes which are connected by two edges of type I and every edge of type III-b connects the same nodes which are connected by an edge of type I, the pair graph is loop-free. Thus the machine M is $\{\frac{1}{2}n(n-1)+1\}$-lossless machine. Q.E.D.

We can show that any n-state k-lossless sequential machine satisfying the upper bound of k is a minimal machine.

Assume that there exists an n-state k-lossless sequential machine satisfying the upper bound of k which is not minimal; then we can obtain an equivalent n'-state sequential machine ($n' < n$). As these two machines realize the same input-output relation, both are $\{\frac{1}{2}n(n-1)+1\}$-lossless. Thus there exists an n'-state $\{\frac{1}{2}n(n-1)+1\}$-lossless sequential machine, which contradicts to (4).

For the lower bound, it is obvious that for any positive integers n, p and q ($q \leqslant p$), there exists a minimal n-state 1-lossless sequential machine with p input symbols and q output symbols. The following machine is one example:

Let $\{B_1, B_2, \ldots, B_p\}$ be a partition on Y. The next-state function is defined as follows:

$$x \in X, \qquad s_{i+1} = \delta(s_i, x), \qquad s_1 = \delta(s_n, x).$$

There exist only one state s_i and two inputs x_j and x_k satisfying

$$\lambda(s_i, x_j) \in B_k, \qquad \lambda(s_i, x_k) \in B_j.$$

All other pairs of states and inputs satisfy

$$\lambda(s, x_h) \in B_h .$$

We can conclude that there exist *minimal* n-state sequential machines satisfying the bounds of (4) for any n.

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